Mean Square Stability Analysis of Some Linear Stochastic Systems

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Abstract

Mean square stability analysis of some continuous and discrete time stochastic systems is carried out in this paper. We present a general approach to mean square stability investigation of systems with multiplicative noise and apply presented theory to discretized linear oscillators as often met in Mechanical Engineering. The analysis relies on the spectral theory of positive operators. As one of the results one obtains a simple and efficient criterion to decide the question of stability of equilibria of linear systems. Conclusions for practical usage and preference of numerical methods solving stochastic differential equations (SDEs) with white noise can be drawn too. For illustration and practical meaningfulness, we describe stability domains of stochastic θ–methods in terms of parametric restrictions.

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1 INTRODUCTION

The analysis of stochastic systems with respect to mean square stability of their equilibria has attracted many researchers, see e.g. Kats & Krasovskij [7], Khas'minskij [8], Kozin [10], Morozan [17], Tsarkov [29] or Willems [32]. Such systems occur in a large number of applications as in Physics, Optics or Mechanical Engineering. Often, these systems can generally be written as systems of stochastic differential equations (SDEs) in Itô or Stratonovich form. There stability examinations play an essential role in judgement on qualitative behaviour of natural processes.

The concept of mean square stability is one of the most attractive and feasible ones within the large branch of stability analysis. Due to facilities of modern computers and progress in numerical analysis of stochastic differential equations (SDEs), see e.g. Kloeden et al [9], the interest in mean square stability analysis has come up once again. The basic questions for any numerical algorithm are accuracy and stability. The question of accuracy has been worked out well. For example, see e.g. Kloeden et al [9], Mil'shtein [15] or Pardoux & Talay [19]. However, the question of stability is fairly underdeveloped and still in its very beginning, despite of a number of recent contributions. These contributions exclusively deal with numerical stability analysis with respect to linear test equations in one dimension. For example, see Hernandez & Spigler [5], Mil'shtein [15], Mitsui & Saito [16] or Peterson [20]. All these papers do not consider multi–dimensional systems which have important practical meaning, e.g. oscillators in Mechanical Engineering, see Bachmann et al [3], Lin & Cai [14] or Soong & Grigoriu [27]. It is worth stressing that many multi–dimensional stochastic systems are not reducable to sets of independent scalar equations in view of stability analysis and test equations. It is also apparent that there is an essential difference between stochastic and deterministic systems in this respect. This fact is caused by the complexity of stability analysis for multi–dimensional stochastic systems. An first approach to numerical stability analysis for multi–dimensional stochastic systems can be found in Artemiev [2] or Schurz [23,24,26]. The practical use of such investigations lies in work out of recommendations and selection procedures for more efficient and accurate numerical algorithms solving SDEs. Both authors are able to find at least one class of methods which provide numerically mean square stable solutions, namely stochastic Rosenbrock–methods and θ–methods (a family of implicit Euler methods), respectively. They also obtain sufficient criterions for numerical mean square stability.

However, there is still a need to search for more efficient criterions to decide the problem of stochastic stability of both continuous and discrete time systems. The decision problem of mean square stability can be reduced to analysis of some matrix equation in general. This is justified by Lyapunov function techniques, see Khas'minskij [8]. We present some further steps towards clarification of this decision problem in both continuous and discrete time setting. The main aim of our work is to find a more practical (efficient) criterions for decision on mean square stability. Some parametric criterions for certain classes of continuous time systems have already been suggested in Levit & Yakubovich [13], Nevelson & Khas'minskij [18], Ryashko [21,22] or Willems [32]. For the sake of illustration with practical meaningfulness, we select the class of linear stochastic oscillators and its discretization by stochastic θ–methods. For deterministic θ–methods, see Stewart & Peplow [28]. The analysis finally results in efficient computation of mean square stability domains of these discrete methods. The whole theory relies on exploitation of spectral theory of positive linear operators which is well understood nowadays, see Krasnosel'skij et al [11].

The paper is organized as follows. In section 2 we carry out some analysis of linear continuous time stochastic systems with respect to mean square stability of their equilibria. For the sake of classification, the notion of mean square equivalence is introduced for stochastic systems. We expose the idea of reduction of number of noise sources of original systems to systems with single noise. An efficient criterion for the decision on mean square stability of continuous time systems is given by Theorem 2. There an interesting relation between the stability behaviour of systems with multiplicative noise and systems with additive noise also comes up. The key idea of presented analysis – the computation of spectral radius of positive operators to decide mean square stability – is outlined in section 2 and following ones. Taking advantage of related theory one finds parametric criterions for mean square stability analysis in more than one dimension. Eventually we illustrate the theory with the class of linear stochastic oscillators with single degree of freedom and multiplicative white noise under its discretization by θ–methods. In section 3 mean square stability analysis for continuous time oscillators is carried out. The examination leads to description of related stability domains. Section 4 extends the presented theory to discrete time stochastic systems. In section 5 the two–parametric family of stochastic θ–methods is introduced in particular for discretization of linear oscillators. Two representatives of this class are investigated with respect to numerical mean square stability in detail. These are the well–known Euler method and an explicit–implicit method where latter method has no counterpart in one–dimensional situation. We express restrictions on step size in terms of oscillator parameters like intensities of stiffness and friction. For the sake of completeness and comparison, some illustration for the classical one–dimensional test equation is also added. The final investigation leads to computation and visualization of corresponding stability domains. The paper is closed by conclusions and further remarks in section 6.

2 MEAN SQUARE STABILITY OF CONTINUOUS TIME SYSTEMS

Stochastic systems can frequently be written as systems of stochastic differential equations (SDEs) driven by independent processes with independent increments. Consider autonomous linear stochastic systems

$$dX(t) \; = \; A\,X(t)\,dt + \sum_{j=1}^{m} B^j\,X(t)\;dW^j(t) \tag{2.1}$$

where $X(t)$ denotes the d–dimensional solution, $A, B^j (j = 1, 2, ..., m)$ real–valued matrices and W^j are uncorrelated standard Wiener processes. In contrast to deterministic integration, the solution of these SDEs strongly depends on the choice of the integration calculus in (2.1). Without loss of generality, we will only take into consideration the well–known Itô interpretation for the corresponding stochastic integration. In passing we note that the different stochastic integral interpretations can be transformed into each other in a natural way, cf. Arnold [1]. Now, recall definition of exponential mean square stability of such systems.

Definition 1. The solution $x \equiv 0$ of system (2.1) is called *exponentially stable in the mean square sense* or shorter *EMS–stable* if there exist constants $\alpha > 0, L > 0$ such that

$$\mathbb{E}\,\|X(t)\|^2 \;\leq\; L\;\exp(-\alpha\,t)\;\mathbb{E}\,\|X(0)\|^2 \tag{2.2}$$

for any $X(0)$ and any $t \geq 0$.

Remark. Throughout this paper, for convenience, we call a system EMS–stable if it has an EMS–stable null solution.

EMS–stability of systems (2.1) has been considered by many authors, see Kats & Krasovskij [7], Levit & Yakubovich [13], Nevelson & Khas'minskij [18], Ryashko [21], [22] or Willems [32]. Common criterions are based on Lyapunov function techniques, see Khas'minskij [8]. In the autonomous case, these techniques lead to the decision problem on positive solvability of matrix equation

$$AV + VA^T + \sum_{j=1}^{m} B^j V B^{j^T} = -C . \tag{2.3}$$

However, this decision problem does not provide practical criteria. For some classes of systems, effective criteria could be found by authors mentioned above. From our view point it still needs to further clarify and simplify the obtained criteria.

2.1 Mean square equivalent systems and noise reduction

The criterions to decide EMS–stability have the most simple and constructive form for n–th order Itô equation. Consider

$$y^{(n)} + (a_1 + \alpha_1 \xi_1)\, y^{(n-1)} + \ldots + (a_n + \alpha_n \xi_n)\, y = 0 \tag{2.4}$$

where $\alpha_i \in \mathbb{R}$ are intensities of parametric noise $W^j = \int_0^t \xi_j(s)ds$. Take $a_i \in \mathbb{R}$. Some stability analysis for system (2.4) has been carried out by Nevelson & Khas'minskij [18]. Using Routh–Hurwitz criterion gives a set of n inequalities which still is fairly laborious to evaluate for large systems.

We are aiming to simplify the analysis of system (2.4) with respect to mean square stability. The obvious change of variables $Y_1 = y$, $Y_2 = y^{(1)}$, ..., $Y_n = y^{(n-1)}$ puts system (2.4) in the form of (2.1) with $d = n = m$,

$$A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ . & . & . & \ldots & . \\ -a_n & -a_{n-1} & -a_{n-2} & \ldots & -a_1 \end{pmatrix}, \; B^j = e\, q_j^T , \; e = \begin{pmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 1 \end{pmatrix} \text{ and}$$

$$[q_j]_i = \begin{cases} 0 & , \; i \neq j \\ -\alpha_{n-j+1} & , \; i = j \end{cases} . \tag{2.5}$$

The simplicity of stability analysis of these systems is connected with the possibility to reduce the number of noise sources W^j. This idea leads to the introduction of a new class of d–dimensional SDEs

$$dX(t) = A\, X(t)\, dt + \sqrt{X^T(t)\, Q\, X(t)}\; \varphi\, dW(t) \tag{2.6}$$

with single Wiener noise W and appropriate positive–semidefinite $d \times d$ matrices Q. φ represents some vector in $\mathbb{R}^d$. Systems of type (2.6) are treated in Ryashko [22]. For justification of this class we introduce the definition of mean square equivalent systems.

Definition 2. Two stochastic processes X, Y are called *mean square equivalent* if their mean square evolutions coincide, i.e.

$$\forall t \geq 0 \qquad \mathbb{E}\, X(t)\, X^T(t) \;=\; \mathbb{E}\, Y(t)\, Y^T(t)\,.$$

Theorem 1. *Assume that* $\mathbb{E}\, X(0)\, X^T(0) \;=\; \mathbb{E}\, Y(0)\, Y^T(0)$.
Then process Y satisfying (2.1) with (2.5) is mean square equivalent with process X governed by (2.6) with

$$Q = \sum_{j=1}^{m} q_j\, q_j^T\,, \quad \varphi = e\,.$$

Remarks. The proof of Theorem 1 is an easy application of Itô formula to systems (2.1) and (2.6), hence it can be omitted here. For the proof, the correlation between W and W^j is not essential. It is also worth noting that the conclusion of Theorem 1 is also valid for any matrix A, but the structure of noise terms B^j has to be specified.

2.2 A criterion of EMS–stability of system (2.6)

The following interesting relation to systems with additive noise comes up. Along with system (2.6) with single multiplicative noise, consider $\mathbb{R}^d$–valued systems

$$dX(t) \;=\; A\, X(t)\, dt \,+\, \varphi\, dW(t) \tag{2.7}$$

with single additive noise and

$$dX(t) \;=\; A\, X(t)\, dt \tag{2.8}$$

without any random perturbation (i.e. deterministic). It is worth noting that, for asymptotically stable systems (2.8), there exists a limit matrix

$$M \;=\; \lim_{t \to +\infty} M_t, \quad M_t \;=\; \mathbb{E}\; X(t)\, X^T(t) \tag{2.9}$$

where $X(t)$ is a solution of (2.7). Moreover, matrix M is a solution of equation

$$A\, M \,+\, M\, A^T \,+\, \varphi\, \varphi^T \;=\; \mathcal{O} \tag{2.10}$$

where $\mathcal{O}$ represents the zero matrix in $\mathbb{R}^{d\times d}$. Now one encounters with the following result.

Theorem 2. *The system (2.6) with single multiplicative noise is EMS–stable if and only if it holds that*
(a) deterministic system (2.8) is asymptotically stable and
(b) system (2.7) with single additive noise has matrix M of stationary second moments satisfying

$$tr(M\,Q) \;<\; 1 \tag{2.11}$$

where $tr(.)$ denotes the trace of inscribed matrix.

Remarks. The proof of Theorem 2 is given in Ryashko [22]. Combining main assertions of Theorems 1 and 2, one can immediately obtain conclusions in view of mean square stability of original system (2.4). The efficiency of received criterion lies in the practical evaluation of (2.11), whereas requirement (a) obviously represents a necessary condition for mean square stability at all.

2.3 Mean square majorants

The efficiency of criterion given by Theorem 2 is connected with the specific choice of matrices B^j for system (2.4), as indicated with (2.5) before. In general situation, this criterion can efficiently be used too. For this purpose we introduce the notion of mean square majorants. In stating assertions below $\mathbb{I}$ denotes the unit matrix of $\mathbb{R}^{d\times d}$.

Definition 3. The stochastic process X is called *mean square majorant* to stochastic process Y if their mean square evolution satisfies

$$\forall t \geq 0 \qquad \mathbb{E}\, X(t)\, X^T(t) \;\geq\; \mathbb{E}\, Y(t)\, Y^T(t)$$

where the corresponding inequality sign is understood in terms of positive–semidefinite matrices.

Theorem 3. *Assume that $\mathbb{E}\, X(0)\, X^T(0) \;\geq\; \mathbb{E}\, Y(0)\, Y^T(0)$ and process X satisfies*

$$dX(t) \;=\; A\, X(t)\, dt \,+\, \sqrt{X^T(t)\, Q\, X(t)}\; d\hat{W}(t) \tag{2.12}$$

where $\hat{W}$ is a d–dimensional vector of Wiener processes with

$$\mathbb{E}\; d\hat{W}\; d\hat{W}^T \;=\; G\, dt$$

and correlation matrix $G \in I\!R^{d\times d}$.

Then process X with one of the following choices

$$(i) \qquad Q = \sum_{j=1}^{m} {B^j}^T B^j, \; G = 1\!I,$$

$$(ii) \qquad Q = 1\!I, \; G = \sum_{j=1}^{m} B^j {B^j}^T$$

is mean square majorant to process Y governed by (2.1).

Proof (Idea). After calculation and comparison of evolutions of second moments for processes X and Y, the proof reduces to verify

$$C_1 := \sum_{j=1}^{m} B^j M {B^j}^T \le tr(Q M) G =: C_2$$

for any positive–semidefinite matrix $M \in \mathbb{R}^{d\times d}$. This matrix relation is equivalent to require the validity of scalar inequality

$$tr(V C_1) \le tr(V C_2)$$

for any positive–semidefinite matrix V, which turns out to be true. $\diamond$

Remarks. One can find systems of type (2.12) which rule as mean square majorant to any original system (2.1). For systems (2.12), a similar theorem as Theorem 2 is valid. Thanks to Theorem 3 and this new more general variant of Theorem 2, it basicly remains to evaluate condition (b) of Theorem 2 for an efficient mean square majorant system to obtain sufficient conditions for mean square stability of the original system.

3 STABILITY OF CONTINUOUS TIME OSCILLATORS

Consider the stochastic oscillator

$$\ddot{x} + (b + \sqrt{\beta}\xi_2)\dot{x} + (a + \sqrt{\alpha}\xi_1)x = 0 \qquad (3.1)$$

with random perturbation of parameters a (coefficent of stiffness) and b (coefficient of damping, e.g. caused by friction) with intensities α and β, respectively. ξ_1 and ξ_2 are formal derivatives of independent standard Wiener processes W^1 and W^2. The change of variables $y = \dot{x}$ leads to equivalent formulation

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -a x - b y - \sqrt{\alpha}\, x\, \xi_1 - \sqrt{\beta}\, y\, \xi_2 . \end{aligned} \qquad (3.2)$$

This system is mean square equivalent to system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -a\,x - b\,y + \sqrt{\alpha\, x^2 + \beta\, y^2}\,\xi \end{aligned} \tag{3.3}$$

where ξ represents the formal derivative of standard Wiener process W.

For stability analysis motivated by Theorem 2, it needs to consider the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -a\,x - b\,y + \xi \end{aligned} \tag{3.4}$$

with single additive noise ξ and related deterministic system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -a\,x - b\,y\,. \end{aligned} \tag{3.5}$$

For any $a > 0, b > 0$, system (3.5) is asymptotically stable and system (3.4) has stationary second moments with

$$M = \begin{pmatrix} m_1 & m_2 \\ m_2 & m_3 \end{pmatrix}, \; m_1 = \frac{1}{2\,a\,b}, \; m_2 = 0, \; m_3 = \frac{1}{2b}. \tag{3.6}$$

Thanks to Theorem 2, we can describe the structure of mean square stability domain belonging to (3.3), and hence for (3.1) too. The domain of EMS–stability is given by

$$\frac{\alpha}{2\,a\,b} + \frac{\beta}{2\,b} < 1\,. \tag{3.7}$$

This restriction clearly devides the (α, β)–plane into regions of stability and instability. Note that the increase of parameters a and b implies an extension of mean square stability domain of oscillator (3.1). Vice versa, the increase of noise intensities α and β reduces its domain of stability.

4 MEAN SQUARE STABILITY OF DISCRETE TIME SYSTEMS

An analogous analysis to that of section 2 can be carried out for discrete time stochastic systems. Such systems naturally occur in numerical solution of SDEs. Consider d–dimensional discrete time systems

$$x_{n+1} = B\,x_n + \sqrt{x_n^T\,Q\,x_n}\,\varphi\,\xi_n \tag{4.1}$$

where ξ_n represent real–valued uncorrelated random variables with

$$\mathbb{E}\,\xi_n \;=\; 0,\; \mathbb{E}\,\xi_n^2 \;=\; h\,.$$

B and Q are $d \times d$ real–valued matrices, Q is positive–semidefinite, φ a vector in $\mathbb{R}^d$. In addition to (4.1), as in the continuous time case, we introduce the auxilary system

$$x_{n+1} \;=\; B\,x_n \,+\, \varphi\,\xi_n \tag{4.2}$$

containing single additive noise, as well as the related deterministic system

$$x_{n+1} \;=\; B\,x_n\,. \tag{4.3}$$

In case of asymptotical stability of system (4.3), there exist a limit matrix

$$M \;=\; \lim_{n\to+\infty} M_n,\; M_n \;=\; \mathbb{E}\,x_n\,x_n^T \tag{4.4}$$

where x_n follows (4.2). Moreover, matrix M is the solution of equation

$$M \;=\; B\,M\,B^T \,+\, h\,\varphi\varphi^T\,. \tag{4.5}$$

Let us recall the notion of exponential mean square stability for discrete time stochastic systems. Define $\mathcal{T}$ as collection of time points (Some authors call $\mathcal{T}$ as *underlying time scale*). Interpret x_n as value of discrete dynamic system correponding to time $t_n \in \mathcal{T}$.

Definition 4. The solution $x \equiv 0$ of system (4.1) is called *exponentially stable in the mean square sense* or shorter *EMS–stable* if there exist constants $\alpha > 0, L > 0$ such that

$$\mathbb{E}\,\|x_n\|^2 \;\le\; L\,\exp(-\alpha\,t_n)\,\mathbb{E}\,\|x_0\|^2 \tag{4.6}$$

for any x_0 and any $t_n \in \mathcal{T}$.

Remark. Throughout this paper, for convenience, we call a discrete system EMS–stable if it has an EMS–stable null solution.

EMS–stability of discrete time stochastic systems has been considered by many authors, see e.g. Willems [32]. In context of numerical solutions, Artemiev [2] and Schurz [23,24,26] studied the mean square behaviour of certain parametric numerical methods. All in all, there is still a need to find more efficient criterions. Our work is strongly related to systems of the form (4.1). Using the spectral theory of positive operators one gains the following criterion.

Theorem 4. *The system (4.1) with single multiplicative noise is EMS–stable if and only if it holds that*
(a) deterministic system (4.3) is asymptotically stable and
(b) system (4.2) with single additive noise has matrix M of stationary second moments satisfying

$$tr(M\,Q) \;<\; 1\,. \tag{4.7}$$

Proof. First, necessity of conditions (a) and (b) of Theorem 4 for EMS–stability is shown. Let system (4.1) be EMS–stable. Then system (4.3) is asymptotically stable. Define $\mathcal{K}$ as the cone of real–valued positive–definite $d \times d$ matrices. Moreover, we know that, for any matrix $C \in \mathcal{K}$, there exists a real–valued $d \times d$ matrix V which satisfies the equation

$$-\,V \,+\, B^T\,V\,B \,+\, h\,\varphi^T\,V\,\varphi\,Q \;=\; -\,C\,. \tag{4.8}$$

This is a discrete time counterpart of continuous time Lyapunov matrix equation (2.3), see also Willems [32]. Consider operators

$$\mathcal{A}[V] \;=\; -\,V \,+\, B^T\,V\,B\,, \quad \mathcal{S}[V] \;=\; h\,\varphi^T\,V\,\varphi\,Q\,. \tag{4.9}$$

Now equation (4.8) can equivalently be rewritten to

$$\mathcal{A}[V] \,+\, \mathcal{S}[V] \;=\; -\,C\,. \tag{4.10}$$

Obviously, asymptotical stability of deterministic system (4.3) is equivalent with existence of the inverse $\mathcal{A}^{-1}$ of operator $\mathcal{A}$. Operator $\mathcal{A}^{-1}$ is negative, i.e. $-\,\mathcal{A}^{-1}\,[\mathcal{K}] \subset \mathcal{K}$. From (4.10) it follows

$$V \,-\, \mathcal{P}\,[V] \;=\; -\,\mathcal{A}^{-1}\,[C] \tag{4.11}$$

where $\mathcal{P} \;=\; -\,\mathcal{A}^{-1}\,\mathcal{S}$ – as a result of multiplication of two positive operators – is positive too. Therefore it holds $V \,-\, \mathcal{P}\,[V] \in \mathcal{K}$, see (4.11). According to Theorem 16.7 from Krasnosel'skij [11], this fact implies the important inequality

$$\rho\,(\,\mathcal{P}\,) \;<\; 1 \tag{4.12}$$

where $\rho(\mathcal{P})$ is the spectral radius of operator $\mathcal{P}$. It follows from the structure of operator $\mathcal{S}$ that

$$\mathcal{P}\,[V] \;=\; h\,\varphi^T\,V\,\varphi\;D \tag{4.13}$$

where matrix $D = -\mathcal{A}^{-1}[Q]$ is a solution of

$$-D + B^T D B = -Q. \tag{4.14}$$

Let V and ρ be an eigenvector and eigenvalue of operator $\mathcal{P}$, respectively, with

$$\mathcal{P}[V] = h\varphi^T V \varphi D = \rho V.$$

Hence, operator $\mathcal{P}$ has a single eigenvalue

$$\rho = h\varphi^T D \varphi. \tag{4.15}$$

Introduce scalar product $< V, D > := tr(V D)$ and conjugate operator $\mathcal{A}^*$ for operator $\mathcal{A}$. We have

$$\mathcal{A}^*[V] = -V + B V B^T.$$

Eventually, relations (4.14) and (4.15) imply

$$\begin{aligned}\rho &= h < D, \varphi\varphi^T > \\ &= h < -\mathcal{A}^{-1}[Q], \varphi\varphi^T > = h < Q, -(\mathcal{A}^*)^{-1}[\varphi\varphi^T] > = < Q, M >\end{aligned}$$

where matrix M solves equation (4.5). This identity together with (4.12) gives inequality (4.7).

Now we check the sufficiency of conditions (a) and (b) for EMS–stability. From previous argumentation it is known that requirements (a) and (b) of Theorem 4 imply the existence of positive operator $\mathcal{P}$ as well as validity of inequality (4.12). Hence, equality

$$(I - \mathcal{P})^{-1} = \sum_{k=0}^{+\infty} \mathcal{P}^k$$

holds, where I represents the identity operator. Obviously, operator $(I - \mathcal{P})^{-1}$ is positive too. It means that, for any $C \in \mathcal{K}$, there exist

$$V = (I - \mathcal{P})^{-1}[-\mathcal{A}^{-1}[C]] \in \mathcal{K}.$$

Therefore matrix $V \in \mathcal{K}$ is a solution of equation (4.8). Consequently, system (4.1) is EMS–stable. $\diamond$

Remarks. An analogous concept of mean square majorants for discrete stochastic systems as in the case of continuous time analysis of section 2 can be introduced. The extension of presented results to vector–valued noise is possible in a straight forward way. We leave such work to the interest of readership.

5 STABILITY OF DISCRETIZED STOCHASTIC OSCILLATORS

In the following we want to study the mean square stability behaviour of continuous time oscillators under discretization. There is a plenty of different stochastic discretization techniques for SDEs, see [9], [15] or [19]. Instead of these contributions we suggest to consider a generalization of deterministic θ–methods, cf. Stewart & Peplow [28].

5.1 Stochastic θ–methods and their convergence

The generalization of deterministic θ–methods is done in two directions. One is to introduce implicitness of different degrees in each components of numerical solution. The other deals with carrying them over to stochastic systems. For mean square stability, it suffices to correct the drift influence on the dynamics in an implicit way. Consider autonomous $\mathbb{R}^d$–valued Itô systems of the form

$$dX(t) \;=\; a(X(t))\,dt \;+\; \sum_{j=1}^{m} b^j\,(X(t))\,dW^j(t) \tag{5.1}$$

where W^j represent independent, identically distributed, standard Wiener processes as above. Define Y_n as value of numerical solution at time t_n, along time–discretization

$$\tau^{\Delta}([0,T]) = \{t_0, t_1, ..., t_{n_T} : 0 \le t_0 \le t_1 \le ... \le t_{n_T} \le T\}$$

of given interval $[0,T]$ with maximum step size $\Delta = \max\{t_{i+1} - t_i : i = 0,1,...,n_T - 1\}$. Then the family of **stochastic θ–methods** is governed by scheme

$$Y_{n+1} \;=\; Y_n \;+\; \Big(\Theta\, a(Y_{n+1}) \;+\; \hat{\Theta}\, a(Y_n)\Big)\,\Delta_n \;+\; \sum_{j=1}^{m} b^j(Y_n)\,\triangle W_n^j \tag{5.2}$$

where $\Delta_n = t_{n+1} - t_n$, $\triangle W_n^j = W^j(t_{n+1}) - W^j(t_n)$ and $\Theta, \hat{\Theta}$ are diagonal matrices with entries $\theta_i, \hat{\theta}_i \in (-\infty, +\infty)$ such that $\Theta + \hat{\Theta}$ coincides with the unit matrix $\mathbb{I}$ of $\mathbb{R}^{d\times d}$. Thus these methods are characterized by vectors $\theta = (\theta_i)$ which determine the degree of implicitness in components of numerical solution, respectively. Of course, the case $\theta_i = 0$ $(\forall i)$ reduces them to well–known Euler method, and the case $\hat{\Theta} \equiv \mathcal{O}$ to implicit Euler method. For further examples, see next subsection.

Let us discuss their mean square convergence and related convergence orders. For this purpose we assume that both exact solution of (5.1) and numerical solution (5.2) are established on one and the same probability space. The **criterion of numerical mean square convergence** is given by the following. There exists real constant

$K = K(T, a, b^j, X(0)) > 0$ such that

$$\sup_{t_n \in \tau^{\Delta}([0,T])} \mathbb{E} \, \|X(t_n) - Y_n\|^2 \leq K \, \Delta^{2\gamma} \tag{5.3}$$

for all $\Delta > 0$ which are sufficiently small. The real constant $\gamma \geq 0$ is called the *global order of mean square convergence* of numerical method for Y. The analysis with respect to this convergence criterion provides the following result.

Theorem 5. *Assume that coefficients a, b^j of SDE (5.1) satisfy the requirements of global Lipschitz continuity and of linear–polynomial growth. In addition let $a \in C^1(\mathbb{R}^d)$ and $\mathbb{E} \, \|X(0)\|^2 < +\infty$.*
Then numerical solution Y governed by (5.2) with $X(0) = Y(0)$ are mean square converging at least with global order $\gamma = 0.5$. Furthermore they possess local order $p_1 = 2.0$ of numerical mean convergence and local order $p_2 = 1.0$ of numerical mean square convergence.

Proof (Sketch). Let $X_{t,x}(t+h)$ and $Y_{t,x}(t+h)$ be the one–step values of exact and numerical solutions at time $t+h$ started at time $t \in [0,T)$, respectively. Verify that

$$\|\mathbb{E} \, (X_{t,x}(t+h) - Y_{t,x}(t+h))\| \leq K_1 \, (1 + \|x\|^2)^{1/2} \, h^{p_1} \text{ and}$$

$$\mathbb{E} \, \|X_{t,x}(t+h) - Y_{t,x}(t+h))\|^2 \leq K_2 \, (1 + \|x\|^2) \, h^{2p_2}$$

for sufficiently small step sizes $h < 1$. Apply Theorem 1 of Mil'shtein [15] to obtain desired orders of numerical convergence. $\diamond$

5.2 Discretizations of stochastic oscillators

Now we are going to apply the numerical methods presented before to stochastic oscillators with one degree of freedom as discussed in sections 2 and 3. Consider the two parametric class of discretization methods

$$\begin{aligned} x_{n+1} &= x_n + [\theta_1 \, f(x_{n+1}, y_{n+1}) + (1-\theta_1) \, f(x_n, y_n)] \, \Delta_n + \psi(x_n, y_n) \, \triangle W_n^1 \\ y_{n+1} &= y_n + [\theta_2 \, g(x_{n+1}, y_{n+1}) + (1-\theta_2) \, g(x_n, y_n)] \, \Delta_n + \sigma(x_n, y_n) \, \triangle W_n^2 \end{aligned} \tag{5.4}$$

applied to two–dimensional continuous time system

$$\begin{aligned} \dot{x} &= f(x,y) + \psi(x,y) \, \xi_1 \\ \dot{y} &= g(x,y) + \sigma(x,y) \, \xi_2 \end{aligned} \tag{5.5}$$

where W^1, W^2 represent two independent standard Wiener processes with their formal derivatives ξ^1, ξ^2, repectively. Numerical methods (5.4) belong to the class of θ–methods with

$$\Theta \;=\; diag(\theta_1, \theta_2)\,.$$

The discretization scheme of oscillator (3.3) using methods (5.4) simplifies to

$$\begin{aligned} x_{n+1} &= x_n + [\theta_1\, y_{n+1} + (1-\theta_1)\, y_n]\,\Delta_n \\ y_{n+1} &= y_n - [\theta_2\,(a\,x_{n+1} + b\,y_{n+1}) + (1-\theta_2)\,(a\,x_n + b\,y_n)]\,\Delta_n + \rho(x_n, y_n)\,\triangle W_n \end{aligned} \tag{5.6}$$

where $\rho(x,y) \;=\; \sqrt{\alpha\, x^2 + \beta\, y^2}$. Obviously, system (5.6) can explicitly be written in vector notation

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \rho(x_n, y_n) \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \triangle W_n \tag{5.7}$$

with $b_{11} \;=\; [1 + \theta_2 bh - \theta_1(1-\theta_2)ah^2]/\delta$, $b_{12} \;=\; [1 + (\theta_2 - \theta_1)bh]h/\delta$, $b_{21} \;=\; -ah/\delta$,

$b_{22} \;=\; [(1-(1-\theta_2)bh) - \theta_2(1-\theta_1)ah^2]/\delta, \varphi_1 \;=\; \theta_1 h/\delta, \varphi_2 \;=\; 1/\delta, \delta \;=\; 1 + \theta_2 bh + \theta_1\theta_2 ah^2$

while using equidistant step size h. System (5.7) is a natural discrete counterpart to continuous time system (3.3). We are interested when this discrete time model preserves the stability property of original continuous time system (3.3), and hence that of (3.1) too.

For sake of completeness, let us recall results on numerical stability of some well–known representatives of θ–methods applied to one–dimensional linear test equation.

5.3 Stability investigation for θ–method in one dimension

Here we report on some results concerning evaluation of stability functions of θ–methods applied to classical one–dimensional test equation which is due to Dahlquist. For this purpose consider one–dimensional continuous time system

$$\dot{x} + (a + \sqrt{\alpha}\,\xi)\,x \;=\; 0 \tag{5.8}$$

with parameters $a > 0, \alpha > 0$ and its discretization by θ–method

$$x_{n+1} \;=\; x_n - a\,[\theta\, x_{n+1} + (1-\theta)\, x_n]\, h - \sqrt{\alpha}\, x_n\, \triangle W_n \tag{5.9}$$

with equidistant step size $h > 0$ and $\theta \geq 0$. ξ represents the formal derivative of underlying standard Wiener process W (i.e. $W(t) = \int_0^t \xi(s)ds$), and $\triangle W_n$ denotes its increment $W(t_{n+1}) - W(t_n)$. After algebraic rearrangements one finds its explicit form

$$x_{n+1} \;=\; \frac{1 \,-\, a\,(1-\theta)\,h \,-\, \sqrt{\alpha}\,\triangle W_n}{1 \,+\, a\,\theta\,h}\; x_n\,. \tag{5.10}$$

It is well–known that inequality

$$\alpha \;<\; 2\,a \tag{5.11}$$

establishes the domain of EMS–stability of system (5.8). The discrete counterpart to (5.11) for EMS–stability of system (5.9) is given by inequality

$$\frac{(1 \,-\, a\,(1-\theta)\,h)^2 \,+\, \alpha\,h}{(1 \,+\, a\,\theta\,h)^2} \;<\; 1 \tag{5.12}$$

which is equivalent to

$$\alpha \;<\; 2\,a \,+\, a^2\,(2\,\theta - 1)\,h\,. \tag{5.13}$$

Four basic conclusions can be drawn by analyzing inequality (5.13) in conjunction with (5.11). First, an enlargement of $\theta \geq 0$ implies monotonically increasing stability domains – the stabilizing effect of θ–methods. Second, for $\theta > 0.5$, discrete time system (5.9) is more stable than corresponding continuous time system (5.8), whereas (5.9) is lesser stable than (5.8) when $\theta < 0.5$. Third, for $\theta = 0.5$, both stability domains coincide! Fourth, for $\theta \geq 0.5$, the increase of drift parameter a leads to monotonic enlargement of corresponding stability domain. All four conclusions hold for any step size $h > 0$!

For the sake of illustration of stability domains corresponding to systems (5.8) and (5.9), we add the plots of figures 1a, 1b and 1c. There the image of stability function

$$f \;=\; f(a,\alpha,\theta,h) \;=\; \alpha \,-\, 2\,a \,-\, a^2\,(2\,\theta - 1)\,h$$

belonging to θ–methods applied to one–dimensional test equation (5.8) is drawn for $\theta \in \{0, 0.5, 1\}$, respectively, while the noise intensity $\alpha = 0.01$ is fixed. For additional convenience, we have also plotted the zero–hyperplane. The sign of the stability function f determines the domain of stability and instability, respectively. That is regions where negative sign of this stability function occurs belong to the domain of stability of related numerical method applied to our one–dimensional test equation. In another words, the domain of stability is given by those parameter values (a, h) where the image of stability function lies beneath of hyperplane $f \equiv 0$.

Figure 1a shows the image of stability function f corresponding to explicit Euler method (i.e. $\theta = 0$). For simplicity in visual comparison, the corresponding stability functions for same parameter regions as in figure 1a are depicted in figures 1b and 1c.

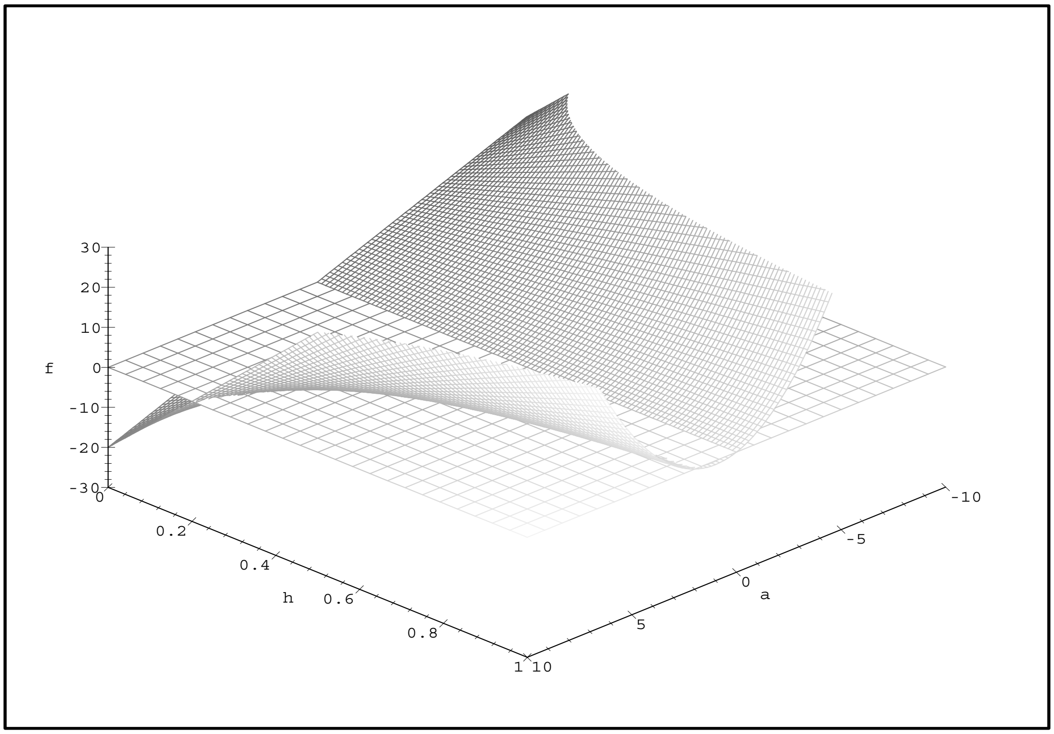

Figure 1a. Stability function of one–dimensional explicit Euler method applied to (5.8) with varying parameter a, varying step size h and constant $\alpha = 0.01$.

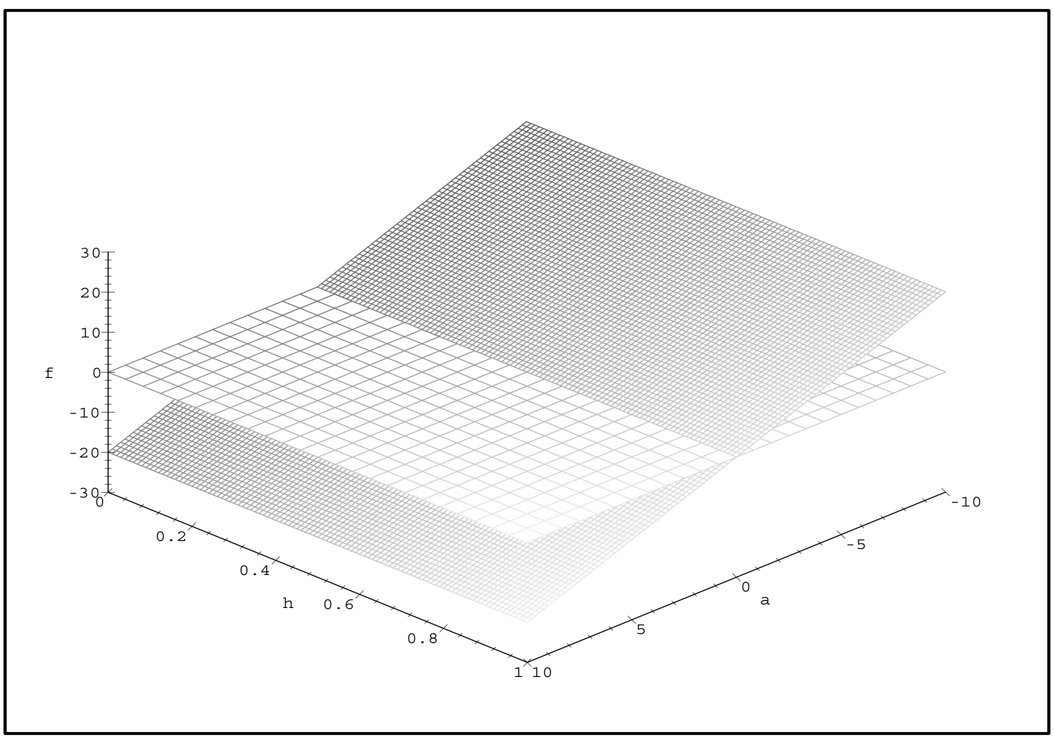

Figure 1b. Stability function of one–dimensional trapezoidal method applied to (5.8) with varying parameter a, varying step size h and constant $\alpha = 0.01$.

One notices an obvious difference between the plots for explicit Euler method (i.e. $\theta = 0$) and implicit methods (i.e. $\theta > 0$). The stability domains are enlarged by using implicit methods (More precisely, one can verify the property of monotonic nesting of stability domains within the class of linear equations!). This is in coincidence with the experience in deterministic numerical analysis. It is worth to mention once again that the stability domain of trapezoidal formula (i.e. $\theta = 0.5$) conincides with that of exact solution of one–dimensional test equation (5.8), see also figure 1b. This fact can be generalized to multi–dimensional linear stochastic systems with both additive and multiplicative noise, cf. Schurz [23,24]. The implicit Euler method (i.e. $\theta = 1$) is the most mean square stable method among θ–methods with $\theta \in [0, 1]$. In general, this fact has been firstly noted in Schurz [23]. For visualization of image of stability function belonging to implicit Euler method, see figure 1c. In principle, one could carry on with enlargement of stability domain by increase of parameter θ while consideration of linear test systems. However, this contradicts to accuracy requirements on numerical methods. Thus it is not advicable to do such an increase. Far more, we recommend to use numerical methods which perform a combination of explicit–implicit methods, as it will be seen in application to stochastic oscillators in the following subsection.

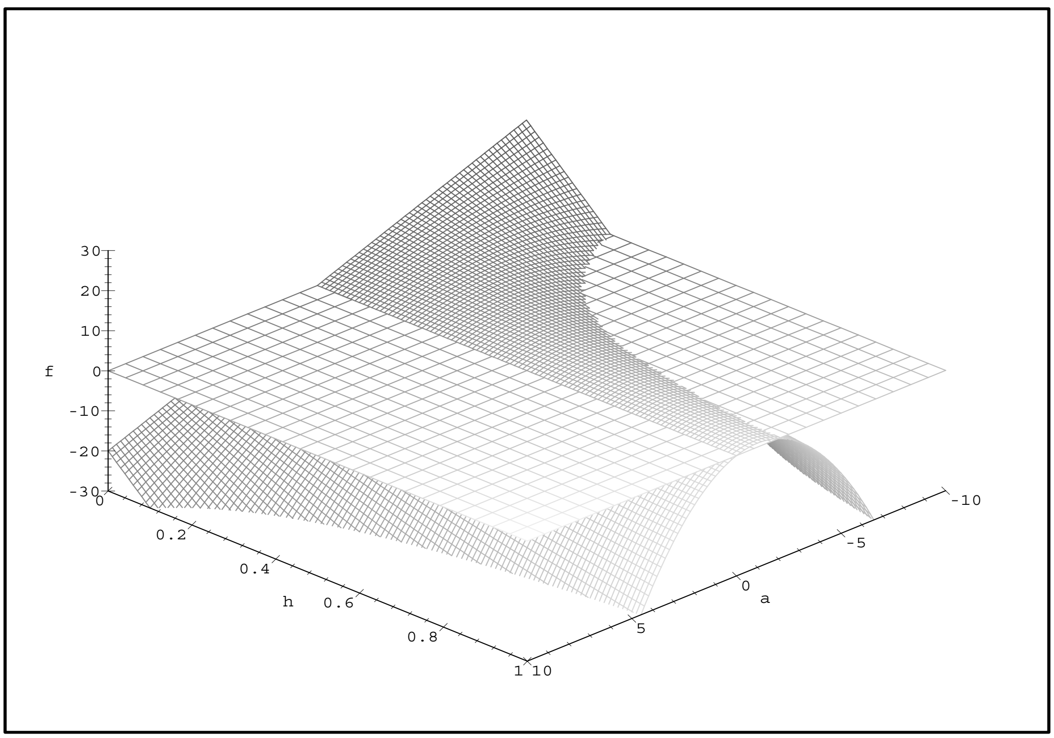

Figure 1c. Stability function of one–dimensional implicit Euler method applied to (5.8) with varying parameter a, varying step size h and constant $\alpha = 0.01$.

5.4 Stability investigations for discretized linear oscillator

Due to the complexity and immense variety of introduction of implicitness in stochastic–numerical methods in higher–dimensional situation, we can only pick up a few algorithms for testing their numerical mean square stability behaviour. The main attraction shall be drawn to fully explicit Euler method and an explicit–implicit method which is a combination of explicit technique in one component and implicit technique in the other component. Let us consider **explicit Euler method** at first. As necessary condition for mean square stability (see Theorem 4), we have to check stability of corresponding deterministic system. Of course, by means of analysis of the related deterministic system, we also obtain conclusions for stability of first moments. For explicit Euler method (i.e. $\theta_1 = \theta_2 = 0$), system (5.7) has the form (4.1) with

$$B = \begin{pmatrix} 1 & h \\ -ah & 1-bh \end{pmatrix}, Q = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \varphi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{5.14}$$

Condition (a) of Theorem 4 (i.e. asymptotical stability of related deterministic system) is equivalent to requirement $|\lambda_i| < 1$ where λ_i are the roots of characteristic polynomial

$$p(\lambda) := det(B - \lambda \mathbb{I}) = \lambda^2 - (2-bh)\lambda + (1-bh) + ah^2 .$$

This condition yields inequalities

$$ah^2 - 2bh + 4 > 0, \quad ah < b. \tag{5.15}$$

As a consequence, one can obtain corresponding stability function

$$f_1 = f_1(a,b,h) = \begin{cases} h - \dfrac{4}{b + \sqrt{b^2 - 4a}} & \text{if} \quad a \leq b^2/4 \\ h - \dfrac{b}{a} & \phantom{\text{if}} \quad a > b^2/4 \end{cases} .$$

The stability domain for related deterministic system is defined by $f_1(a,b,h) < 0$. That is negative sign of stability function f_1 establishes the domain in parameter space (a,b,h) where explicit Euler method has asymptotically stable null solution. The choice of step size for numerical solutions with stable first moments is restricted by the size of parameters a, b. Both small values of b and large values of b lead to usage of smaller step sizes while a is fixed, respectively. The image of stability function f_1 with constant coefficient a is plotted in figure 2a. It is clearly seen that the choice of step size h is strongly connected with parameter b.

Now, continue with numerical stability of second moments of fully explicit Euler method. From Theorem 4 we know that trace–criterion (b) has to be evaluated. Here one finds

$$tr(MQ) = \alpha m_1 + \beta m_3 \tag{5.16}$$

where matrix M as in (3.6) with

$$m_1 = \frac{2 - bh + ah^2}{a(4b - (4a + 2b^2)h + 3abh^2 - a^2h^3)}, \quad m_2 = -\frac{hm_3}{2} \quad \text{and}$$

$$m_3 = \frac{2}{4b - (4a + 2b^2)h + 3abh^2 - a^2h^3}.$$

Thus trace–criterion (b) reduces to requirement

$$\alpha\, m_1(a, b, h) + \beta\, m_3(a, b, h) < 1. \tag{5.17}$$

On the base of this criterion one can construct and investigate the domains of mean square stability of fully explicit Euler method for various parameter values a, b, α, β, h.

For sake of illustration, consider the case of $\alpha = 0$ in more detail. That is random perturbations of stiffness parameter a are absent. Then the stability domain in view of parameter β (intensity of perturbation of friction coefficient b) is defined by

$$\beta < 2b - (2a + b^2)h + \frac{3}{2}abh^2 - \frac{a^2}{2}h^3 = \frac{1}{2}(b - ah)(ah^2 - 2bh + 4). \tag{5.18}$$

As a consequence, the stability domain of fully explicit Euler method is smaller than that of corresponding continuous time oscillator (compare inequality (3.7) with (5.18)). The image of related stability function

$$f_2 = f_2(a, b, \alpha = 0, \beta, h) = \beta - \frac{1}{2}(b - ah)(ah^2 - 2bh + 4)$$

is shown in figure 2b. For visualization of corresponding stability domain, parameters a, α, β are fixed.

Eventually, the numerical stability of an **explicit–implicit method** is investigated for linear oscillators. Consider θ–method (5.4) with $\theta_1 = 0$ and $\theta_2 = 1.0$. Then this method applied to linear oscillator (3.3) possesses a scheme of form (4.1) with

$$B = \begin{pmatrix} 1 & h \\ -\frac{ah}{1 + bh} & \frac{1 - ah^2}{1 + bh} \end{pmatrix}, Q = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \varphi = \begin{pmatrix} 0 \\ \frac{1}{1 + bh} \end{pmatrix}. \tag{5.19}$$

Once again one can apply Theorem 4. For stability of related deterministic system, we analyze characteristic polynomial

$$p(\lambda) := det(B - \lambda \mathbb{I}) = \lambda^2 - (1 + \frac{1 - ah^2}{1 + bh})\lambda + \frac{1}{1 + bh}.$$

The restriction $|\lambda_i| < 1$ on its roots λ_i yields inequality

$$ah^2 - 2bh - 4 < 0. \tag{5.20}$$

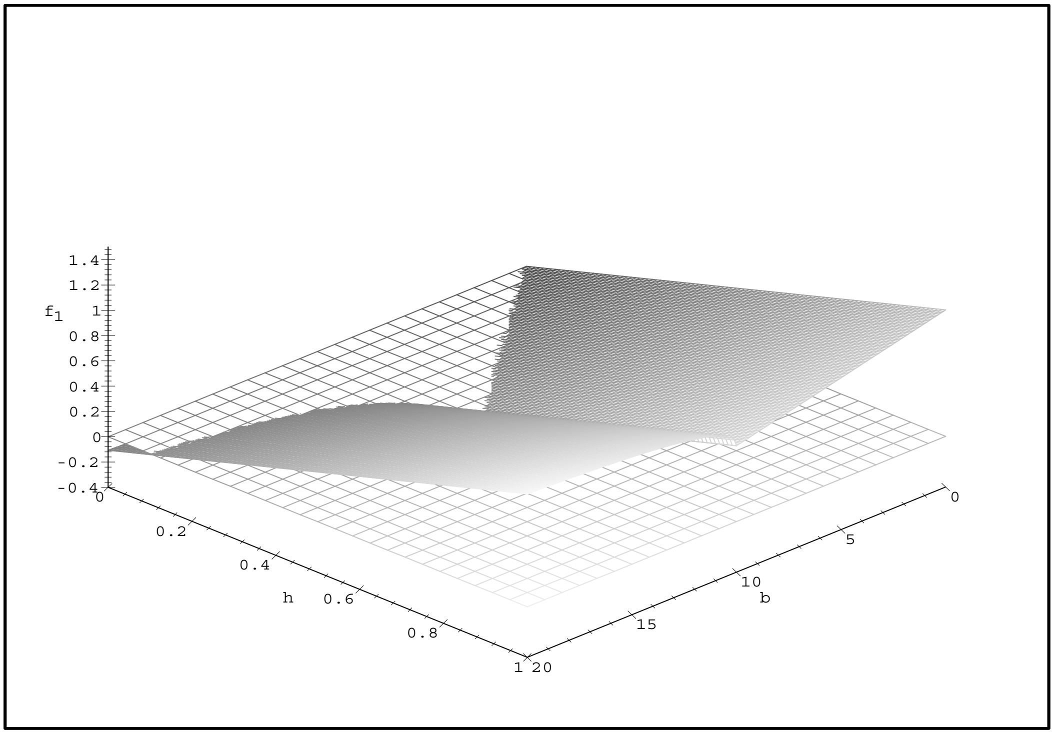

Figure 2a. Stability function of Euler method applied to deterministic linear oscillator (3.3) with varying parameter b, varying step size h and constant $a = 25.0$.

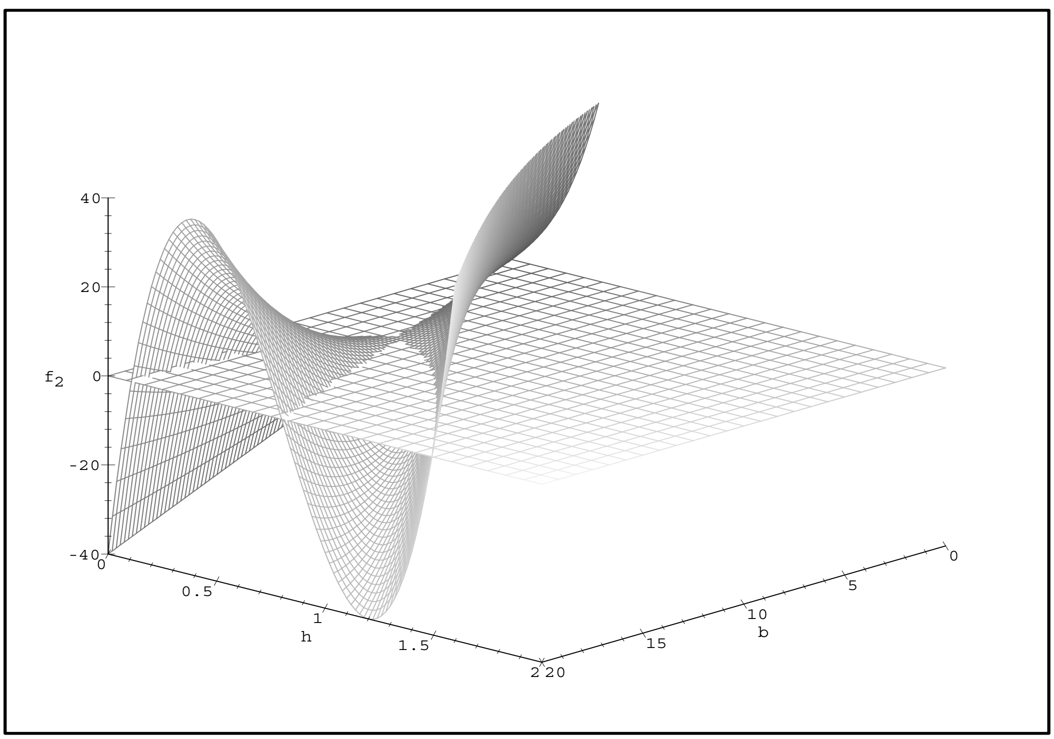

Figure 2b. Stability function of Euler method applied to linear oscillator (3.3) with varying parameter b, varying step size h and constants $a = 25.0, \beta = 0.01$ in the absence of perturbations of stiffness (i.e. $\alpha = 0$).

As a result, one receives the restriction on step size h with

$$h \; < \; \frac{b \, + \, \sqrt{b^2 \, + \, 4\, a}}{a} \, . \tag{5.21}$$

The related stability function

$$f_1 \; = \; f_1(a,b,h) \; = \; h \; - \; \frac{b \, + \, \sqrt{b^2 \, + \, 4\, a}}{a}$$

is visualized in figure 3a. Comparing figures 2a and 3a, there the enlargement of stability domain for deterministic system can be seen while using implicit techniques. Moreover, a simple sufficient condition for asymptotical stability of related deterministic system is

$$h \; < \; \frac{2}{\sqrt{a}} \, .$$

In contrast to analysis of explicit Euler method, this latter restriction does not depend on parameter b. Also the condition on parameter a is less restrictive.

It remains to examine the stability of equation of second moments for the explicit–implicit method with (5.14). The coefficients of matrix M of stationary second moments are

$$m_1 \; = \; \frac{2 \, + \, b\, h}{a\, b\, (4 \, + \, 2\, b\, h \; - \; a\, h^2)} \, , \; m_2 \; = \; -\frac{h\, m_3}{2} \, , \; m_3 \; = \; \frac{2}{b\, (4 \, + \, 2\, b\, h \; - \; a\, h^2)} \, . \tag{5.22}$$

Then, after evaluation of trace–criterion (b) of Theorem 4, one finds the restriction on the step size h given by

$$h \; < \; \frac{b \, - \, \frac{\alpha}{2\, a} \, + \, \sqrt{(b \, - \, \frac{\alpha}{2a})^2 \, + \, a\, (4 \, - \, \frac{2\, \alpha}{a\, b} \, - \, \frac{2\, \beta}{b})}}{a} \, . \tag{5.23}$$

For sake of illustration, we confine further discussion to the case $\alpha = 0$, i.e. the absence of perturbations of stiffness parameter a. In this special case the trace–criterion (b) is equivalent with

$$\beta \; < \; 2\, b \, + \, b^2\, h \; - \; \frac{a}{2}\, b\, h^2 \tag{5.24}$$

which yields stability function

$$f_2 \; = \; f_2(a,b,\alpha=0,\beta,h) \; = \; \beta \; - \; 2\, b \; - \; b^2\, h \, + \, \frac{a}{2}\, b\, h^2$$

for second moments of explicit–implicit method with (5.14). For steps sizes $h \in (0, 2\,\frac{b}{a})$, the stability domain of explicit–implicit method is larger than that of original continuous time system (3.3). Note that fully explicit Euler method is not stable for $h > \frac{b}{a}$. In contrast to fully explicit Euler method, the increase of parameter b (when $b^2 > 4a$)

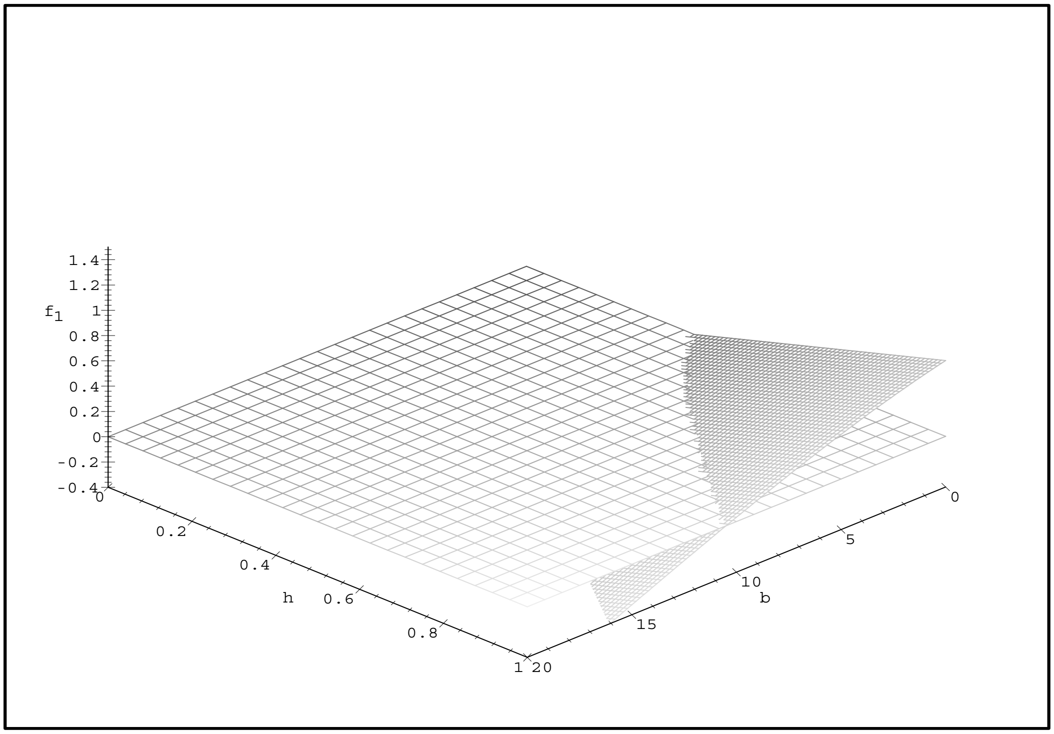

Figure 3a. Stability function of explicit–implicit method applied to deterministic linear oscillator (3.3) with varying parameter b, varying step size h and $a = 25.0$.

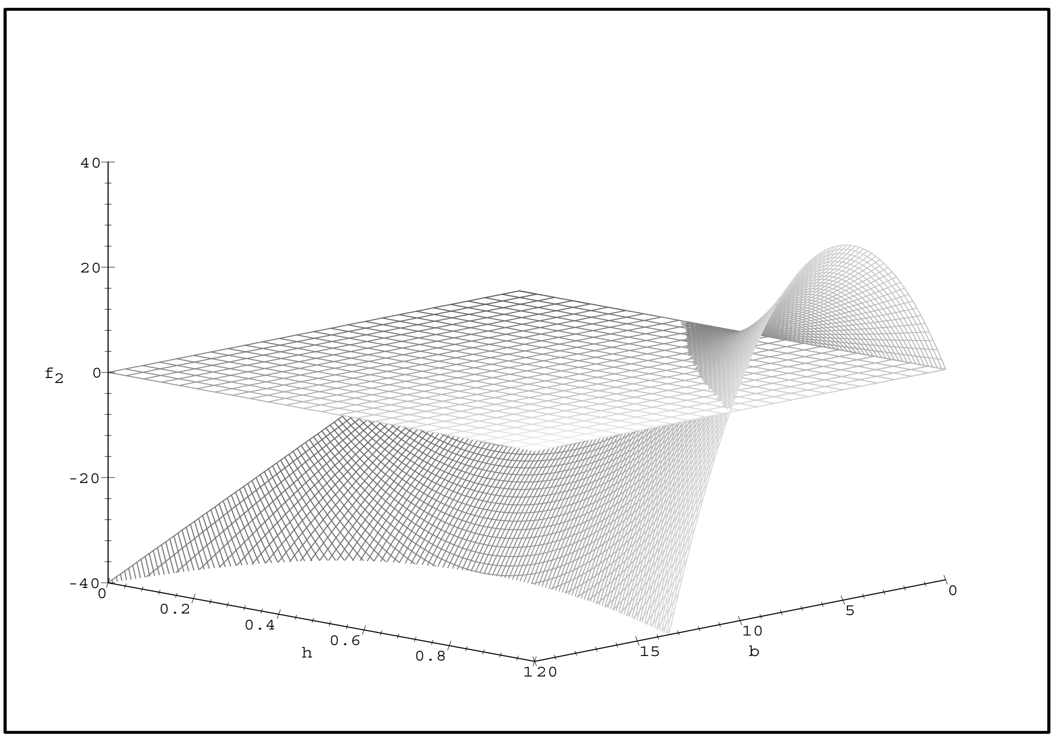

Figure 3b. Stability function of explicit–implicit method applied to linear oscillator (3.3) with varying parameter b, varying step size h and $a = 25.0, \beta = 0.01$ in the absence of perturbations of stiffness (i.e. $\alpha = 0$).

gives an extension of stability domain (which is also true in the case $\alpha > 0$). This is clearly visible in figure 3b. In plotting the related stability function f_2 it occurs another worth mentioning effect of visualization. From figure 3b one might conclude that the considered method has stable second moments even for very small parameters b, while $\beta > 0$ is fixed. This obviously contradicts to inequality (5.24). Such 'misprints' can happen when scaling is done from a very far–distant point of view. Then focussing to critical region (as here the domain where b is sufficiently small) provides visual clarification about the sign of stability function, and hence about stability. This observation is seen in figure 3c. As a consequence, one has to take some care while judging on base of graphic visualization.

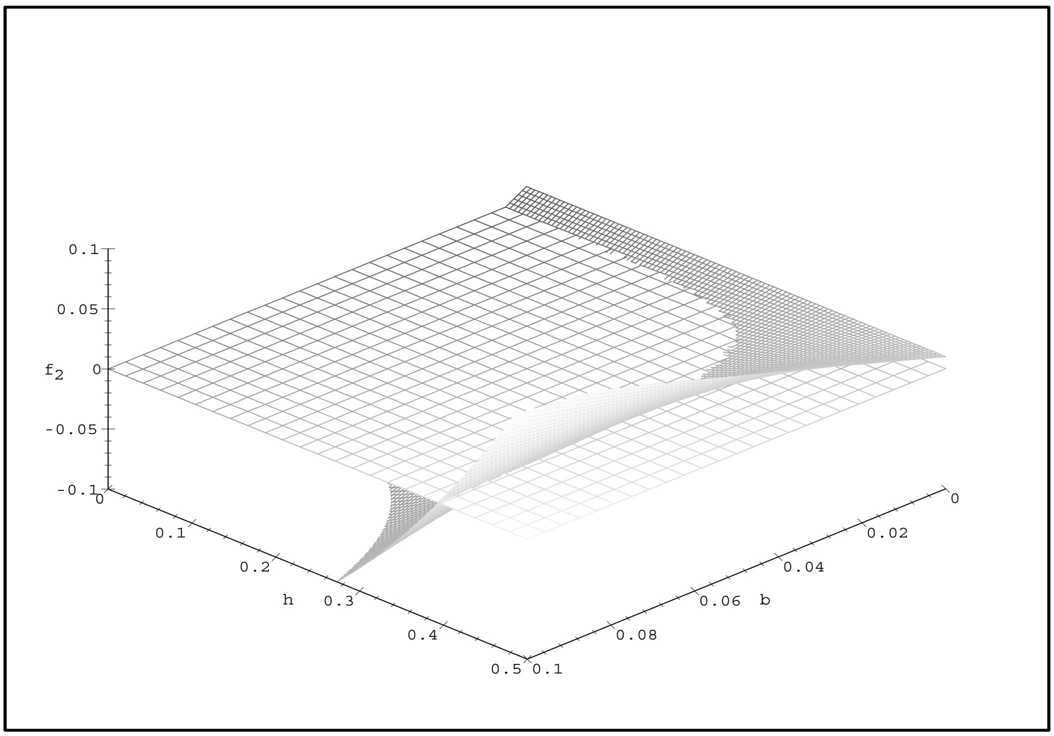

Figure 3c. Stability function of explicit–implicit method applied to linear oscillator (3.3) with varying parameter b, varying step size h and $a = 25.0, \beta = 0.01$ in the absence of perturbations of stiffness (i.e. $\alpha = 0$).

6 CONCLUSIONS AND REMARKS

Some analysis of stochastic systems with respect to mean square stability has been carried out in this contribution. The theoretical approach relies on the spectral theory of positive operators. By means of spectral radius of involved operators we have found practical criteria to classify linear stochastic systems in view of mean square stability, both in discrete and continuous time (cf. Theorems 2 and 4). In some cases it is favourable to consider mean square equivalent or majorant systems, as seen in section 2. For stochastic differential equations (SDEs) we are able to construct mean square stable numerical solutions. Essential differences between possible discretizations can be noticed. All in all, as in deterministic analysis, implicit techniques are favourable in adequate numerical solution of SDEs (see the example of stochastic oscillator in sections 3 and 5). We have introduced and examined the class of stochastic θ–methods. The dynamics of deterministic θ–methods is well understood nowadays, cf. Stewart & Peplow [28]. For example, the implicit trapezoidal and midpoint rule (i.e. $\theta_i = 0.5$) avoid the existence of spurious solutions, or the implicit Euler method (i.e. $\theta_i = 1.0$) is BN–stable for the class of dissipative nonlinear differential equations. In stochastics, for linear oscillators perturbed by multiplicative white noise, appropriate incorporation of implicitness by θ–methods lead to an enlargement of corresponding mean square stability domains. Thus numerical solutions get more and more stabilized while increasing implicitness. However, one has to be very careful while using implicit methods. One aim is to achieve a balance between numerical stability and other qualitative features of numerical solutions. For example, only a small subclass of θ–methods (e.g. that of trapezoidal or midpoint rule) guarantees complete preservation of stationary probabilistic laws of continuous time systems with additive noise. This fact is proved for linear autonomous systems in Schurz [26]. Another effect is met in numerical solutions under algebraic side–conditions. Then explicit methods and some implicit θ–methods are not sufficient to ensure algebraic constraints almost surely. For examples, see Schurz [25]. In the presence of certain multiplicative noise terms one has to incorporate a kind of 'stochastic implicitness'. This is illustrated in [25]. More general methods can also be examined, however in a much more laborious way. Then, mostly one can only provide numerical approximations of corresponding stability functions. This sprinkles the scope of this paper, hence it is omitted here.

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